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Integrable systems of derivative nonlinear Schrödinger type and their multi-Hamiltonian structure

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Abstract

A spectral problem and the associated hierarchy of Schrödinger type equations are proposed. It is shown that the hierarchy is integrable in Liouville's sense and possesses multi-Hamiltonian structure. It is found that several kinds of important equation such as the Kaup–Newell (KN) equation, the Chen–Lee–Liu (CLL) equation, the Gerdjikov–Ivanov (GI) equation, the modified Korteweg–de Vries equation and the Sharma–Tasso–Olever equation are members in the hierarchy as its special reductions. Moreover, KN, CLL and GI equations are described by using a unified generalized derivative Schrödinger equation involving a parameter, and their Hamiltonian structure and Lax pairs are also given by unified and explicit formulae.

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1. Introduction

It is well known that, given a properly chosen spectral problem, one can relate it to a hierarchy of nonlinear equations. A central and very important topic in the study of an integrable system is to search for new Lax or Liouville integrable systems, as many as possible and such that they are associated with certain evolution equations with physical meaning [1–6]. In this paper, we consider a spectral problem

$$y_x = Uy = \begin{pmatrix} -\lambda^2 + \beta qr & \lambda q \\ \lambda r & \lambda^2 - \beta qr \end{pmatrix} y \tag{1}$$

where q, r are two potentials, and β is an arbitrary parameter. This spectral problem is a similar extension of the Kaup–Newell (KN) spectral problem [7, 8]. By setting

$$\begin{aligned} \tilde{y} &= \begin{pmatrix} \exp(-\beta \int qr\,\mathrm{d}x) & 0 \\ 0 & \exp(\beta \int qr\,\mathrm{d}x) \end{pmatrix} y \\ q &= \tilde{q} \exp\left(2\beta \int \tilde{q}\tilde{r}\,\mathrm{d}x\right) \qquad r = \tilde{r} \exp\left(-2\beta \int \tilde{q}\tilde{r}\,\mathrm{d}x\right) \end{aligned} \tag{2}$$

and by simple calculation, we know that the spectral problem (1) is equivalent to the standard KN spectral problem

$$\tilde{y}_x = \begin{pmatrix} -\lambda^2 & \lambda\tilde{q} \\ \lambda\tilde{r} & \lambda^2 \end{pmatrix} \tilde{y}. \tag{3}$$

A derivative nonlinear Schrödinger (DNLS) equation associated with spectral problem (3) is the KN equations [7,8]

$$\begin{aligned} &q_t + q_{xx} + (q^2 r)_x = 0 \\ &r_t - r_{xx} + (qr^2)_x = 0. \end{aligned} \tag{4}$$

To study the effect of higher-order perturbations, another two celebrated DNLS equations also have been proposed and studied for some years, which are the Chen–Lee–Liu (CLL) equations [9,10]

$$\begin{aligned} &q_t + q_{xx} + qrq_x = 0 \\ &r_t - r_{xx} + qrr_x = 0 \end{aligned} \tag{5}$$

and Gerdjikov–Ivanov (GI) equations [11,12]

$$\begin{aligned} &q_t + q_{xx} - q^2 r_x - \tfrac{1}{2} q^3 r^2 = 0 \\ &r_t - r_{xx} - r^2 q_x + \tfrac{1}{2} q^2 r^3 = 0. \end{aligned} \tag{6}$$

These three systems (4)–(6) are usually called DNLSI, DNLSII and DNLSIII equations respectively. It is found that they may be transformed into each other by a gauge transformation, and the method of gauge transformation also can be applied to some generalized cases [13–17]. In recent years, the spectral problem, Hamiltonian structure, Painlevé property, exact solutions and other properties associated with the KN equation have been investigated in detail [7, 8, 14, 15, 18]. Little work has been done on the CLL equation (5) and the GI equation (6), since corresponding results for these two equations may be obtained from the KN equation (4) by some gauge transformation in principle [11,13]. However to obtain their explicit form, we must solve an integrable equation such as (2) in practice [11,13,16]. The integration will become very complicated with increase of iterative times, especially in multi-soliton solutions. This is not convenient for their applications and it is necessary to give their explicit results. In this paper, with the help of the spectral problem (1), we study these three equations in a unified and explicit way. We first derive a hierarchy of DNLS type equations corresponding to the spectral problem (1). Then we show that the hierarchy is integrable in Liouville's sense and possesses multi-Hamiltonian structure by means of trace identity [1,2]. It is interesting that several kinds of important equation such as the KN equation, the CLL equation, the GI equation, the modified KdV equation and the Sharma–Tasso–Olver (STO) equation [19–21] belong to the hierarchy as special reductions. In this way, the KN, CLL and GI equations can be described by using a generalized derivative Schrödinger equation with a parameter. Moreover, their Hamiltonian structure and Lax representation are also established by unified and explicit formulae.

2. The hierarchy of equations and its Hamiltonian structure

We first solve the adjoint representation of spectral problem (1)

$$V_x = [U, V] = UV - VU$$

with

$$V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \lambda^{-j}$$

and obtain the following recursive formulae:

$$\begin{aligned}
&a_{2j+1} = b_{2j} = c_{2j} = 0\\
&a_{2jx} = qc_{2j+1} - rb_{2j+1} = \beta qr(qc_{2j-1} - rb_{2j-1}) + \tfrac{1}{2}(qc_{2j-1x} + rb_{2j-1x})\\
&b_{2j+1} = -\tfrac{1}{2}b_{2j-1x} - qa_{2j} + \beta qrb_{2j-1}\\
&c_{2j+1} = \tfrac{1}{2}c_{2j-1x} - ra_{2j} + \beta qrc_{2j-1}.
\end{aligned}$$

The above recursion equations can be solved successively to deduce that

$$a_0 = -2 \qquad b_1 = 2q \qquad c_1 = 2r$$

$$a_2 = qr \qquad b_3 = -q_x + (2\beta - 1)q^2 r$$

$$c_3 = r_x + (2\beta - 1)qr^2 \qquad a_4 = \tfrac{1}{2}(qr_x - rq_x) + \tfrac{1}{4}(8\beta - 3)q^2r^2$$

$$b_5 = \tfrac{1}{4}[2q_{xx} - 4\beta q^2 r_x - 6(2\beta - 1)qrq_x + (8\beta^2 - 12\beta + 3)q^3r^2] \tag{7}$$

$$c_5 = \tfrac{1}{4}[2r_{xx} + 4\beta r^2 q_x + 6(2\beta - 1)qrr_x + (8\beta^2 - 12\beta + 3)q^2r^3] \tag{8}$$

$$a_6 = \tfrac{1}{8}[2qr_{xx} + 2rq_{xx} - 2q_xr_x + 6(2\beta - 1)qr(qr_x - rq_x) + (24\beta^2 - 24\beta + 5)q^3r^3] \tag{9}$$

and

$$\begin{pmatrix} c_{2j+1} \\ b_{2j+1} \end{pmatrix} = L_1L_2 \begin{pmatrix} c_{2j-1} \\ b_{2j-1} \end{pmatrix} \qquad j = 1, 2, \ldots \tag{10}$$

where

$$L_1 = -\frac{1}{2}\begin{pmatrix} r\partial^{-1}r & -1 + r\partial^{-1}q \\ 1 + q\partial^{-1}r & q\partial^{-1}q \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & \partial - 2\beta qr \\ \partial + 2\beta qr & 0 \end{pmatrix}$$

are two skew-symmetric operators; that is, $L_1^* = -L_1$, $L_2^* = -L_2$.

Consider the auxiliary problem

$$y_t = V^{(n)}y \tag{11}$$

where

$$V^{(n)} = \begin{pmatrix} \Delta_n & 0 \\ 0 & -\Delta_n \end{pmatrix} + \sum_{j=0}^{n} \begin{pmatrix} a_{2j}\lambda^{2(n-j)+2} & b_{2j+1}\lambda^{2(n-j)+1} \\ c_{2j+1}\lambda^{2(n-j)+1} & -a_{2j}\lambda^{2(n-j)+2} \end{pmatrix}.$$

Then the compatibility condition between (1) and (11) gives the zero-curvature equation $U_t - V_x^{(n)} + [U, V^{(n)}] = 0$; that is,

$$\begin{aligned}
&\beta(qr)_t = \Delta_{nx}\\
&q_t = b_{2n+1x} - 2\beta qrb_{2n+1} + 2q\Delta_n\\
&r_t = c_{2n+1x} + 2\beta qrc_{2n+1} - 2r\Delta_n
\end{aligned}$$

from which we can obtain

$$\Delta_n = \beta\partial^{-1}(qc_{2n+1x} + rb_{2n+1x}) + 2\beta^2\partial^{-1}qr(qc_{2n+1} - rb_{2n+1}) = 2\beta a_{2(n+1)}$$

and a hierarchy of evolution equations

$$\begin{aligned}
\begin{pmatrix} q_t \\ r_t \end{pmatrix} &= L_3L_2 \begin{pmatrix} c_{2n+1} \\ b_{2n+1} \end{pmatrix} = (L_3L_2)(L_1L_2)\begin{pmatrix} c_{2n-1} \\ b_{2n-1} \end{pmatrix}\\
&= (L_3L_2)(L_1L_2)^n \begin{pmatrix} 2r \\ 2q \end{pmatrix} \qquad n = 1, 2, \ldots
\end{aligned} \tag{12}$$

where

$$L_3 = \begin{pmatrix} 1 + 2\beta q\partial^{-1}r & 2\beta q\partial^{-1}q \\ -2\beta r\partial^{-1}r & 1 - 2\beta r\partial^{-1}q \end{pmatrix}.$$

In the following we will establish the Hamiltonian structure for the hierarchy (12) and show they are integrable in Liouville's sense. In order to apply the trace identity [1, 2], we need to rewrite (12) in another form. We introduce

$$G_{2j+1} = (c_{2j+1} + 2\beta r a_{2j}, b_{2j+1} + 2\beta q a_{2j})^T.$$

Noting that $a_{2j} = \partial^{-1}(qc_{2j+1} - rb_{2j+1})$, we have

$$(c_{2j+1}, b_{2j+1})^T = L_3^* G_{2j+1} \tag{13}$$

where L_3^* is a conjugation operator of L_3. In this way, the hierarchy (12) can be rewritten in the form

$$u_t = JG_{2n+1} = JLG_{2n-1} = JL^n G_1 \tag{14}$$

where $u = (q, r)^T$, $J = L_3 L_2 L_3^*$, $L = L_3^{*-1} L_1 L_2 L_3^*$.

Proposition 1. *JL^k $(k = 0, 1, 2, \ldots, n)$ are all skew-symmetric operators.*

Proof. Since L_1 and L_2 are skew symmetric, it is clear that J and $JL = L_3 L_2 L_1 L_2 L_3^*$ are skew symmetric. Suppose that JL^{k-1} is skew symmetric, then it holds that

$$(JL^k)^* = (JL^{k-1}L)^* = L^*(JL^{k-1})^* = -L^* JL^{k-1} = L^* J^* L^{k-1} = -JLL^{k-1} = -JL^k$$

which implies that JL^k is skew symmetric. The proof is completed. □

Following the notation used in [1, 2], we take that the Killing–Cartan form $\langle A, B\rangle$ is $\mathrm{tr}(AB)$. Then direct calculation gives

$$\left\langle V, \frac{\partial U}{\partial q}\right\rangle = c\lambda + 2\beta ra \qquad \left\langle V, \frac{\partial U}{\partial r}\right\rangle = b\lambda + 2\beta qa$$

$$\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = -4a\lambda + br + cq.$$

By using the trace identity, we have

$$\frac{\delta}{\delta u}(-4a\lambda + br + cq) = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}(\lambda^{\gamma}(c\lambda + 2\beta ra, b\lambda + 2\beta qa)^T). \tag{15}$$

Substituting

$$a = \sum_{n\geqslant 0} a_{2n}\lambda^{-2n} \qquad b = \sum_{n\geqslant 0} b_{2n+1}\lambda^{-2n-1} \qquad c = \sum_{n\geqslant 0} c_{2n+1}\lambda^{-2n-1}$$

into equation (15) leads to

$$\frac{\delta}{\delta u}(-4a_{2n+2} + rb_{2n+1} + qc_{2n+1}) = (-2n + \gamma) G_{2n+1}(\bar{c}_{2n+1}, \bar{b}_{2n+1})^T. \tag{16}$$

To fix the γ, we let $n = 0$ in (16) and find $\gamma = 0$. Therefore we conclude that

$$G_{2n+1} = \frac{\delta H_n}{\delta u} \tag{17}$$

where

$$H_0 = 2qr \qquad H_n = \frac{4a_{2n+2} - rb_{2n+1} - qc_{2n+1}}{2n} \qquad n \geqslant 1. \tag{18}$$

Combining (14) with (17) gives the desired multi-Hamiltonian structure of the generalized KN hierarchy (12)

$$u_t = J\frac{\delta H_n}{\delta u} = JL\frac{\delta H_{n-1}}{\delta u} = JL^n\frac{\delta H_0}{\delta u} \qquad n = 1, 2, \ldots. \tag{19}$$

Finally we discuss the integrability of the hierarchy (12) or (19). It is crucial to show the existence of infinite involutive conserved densities. Usually the inner product between two functions f and g is defined by $(f, g) = \int f \cdot g \,\mathrm{d}x$, and the Poisson bracket is defined by $\{f, g\} = (\frac{\delta f}{\delta u}, J\frac{\delta g}{\delta u})$. In particular, f and g are called involutive if $\{f, g\} = 0$.

Proposition 2. *The Hamiltonian functions* $\{H_n\}$ $(n = 0, 1, \ldots)$ *given by (18) constitute common conserved densities for the whole hierarchy (19).*

Proof. By using proposition 1, we find that

$$\{H_n, H_m\} = \left(\frac{\delta H_n}{\delta u}, J\frac{\delta H_m}{\delta u}\right) = (L^n G_1, JL^m G_1) = (L^n G_1, L^* J L^{m-1} G_1)$$
$$= (L^{n+1} G_1, JL^{m-1} G_1) = \{H_{n+1}, H_{m-1}\}.$$

Repeating the above argument gives

$$\{H_n, H_m\} = \{H_m, H_n\} = \{H_{m+n}, H_0\}. \tag{20}$$

On the other hand, we find

$$\{H_m, H_n\} = (L^m G_1, JL^n G_1) = (J^* L^m G_1, L^n G_1) = -\{H_n, H_m\}. \tag{21}$$

Then combining (20) with (21) leads to

$$\{H_m, H_n\} = 0$$

which implies that $\{H_m\}$ are in involution. Furthermore we have

$$\left(\int H_m \, \mathrm{d}x\right)_t = \left(\frac{\delta H_m}{\delta u}, u_t\right) = \left(\frac{\delta H_m}{\delta u}, J\frac{\delta H_n}{\delta u}\right) = \{H_m, H_n\} = 0$$

which shows that $\{H_m\}$ are also conserved densities. The proof is completed. □

In summary, we arrive at the following result.

Theorem 1. *(i) The hierarchy (19) is an integrable Hamiltonian system in the Liouville sense. (ii) The Hamiltonian functions* $\{H_m\}$ *are conserved densities of the whole hierarchy (19) and they are involutive in pairs.*

3. Some important equations in the hierarchy

In the following, we provide some interesting equations that are contained in the hierarchy (19) and give them the explicit formulation of Hamiltonian structure and Lax pairs.

Example 1. The first system of the hierarchy (19) $(n = 1)$ is a set of generalized DNLS equations

$$\begin{aligned} &q_t + q_{xx} - 2(2\beta - 1)qrq_x - (4\beta - 1)q^2 r_x - \beta(4\beta - 1)q^3 r^2 = 0 \\ &r_t - r_{xx} - 2(2\beta - 1)qrr_x - (4\beta - 1)r^2 q_x + \beta(4\beta - 1)q^2 r^3 = 0 \end{aligned} \tag{22}$$

which is a unified expression of KN, CLL and GI equations (4)–(6). They correspond to $\beta = 0$, $\beta = 1/4$ and $\beta = 1/2$ of equations (22) respectively. According to theorem 1, we conclude that system (22) is Liouville integrable and possesses the bi-Hamiltonian structure

$$u_t = J\frac{\delta H_1}{\delta u} = JL\frac{\delta H_0}{\delta u}$$

where Hamiltonian functions H_0 and H_1 are

$$H_0 = 2qr \qquad H_1 = \tfrac{1}{2}[rq_x - qr_x + (4\beta - 1)q^2 r^2]. \tag{23}$$

The Lax pairs corresponding to the system (22) may be given by the spectral problem (1) and the auxiliary problem

$$y_t = V^{(1)} y \qquad V^{(1)} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & -v_{11} \end{pmatrix}$$

with

$$v_{11} = -2\lambda^4 + qr\lambda^2 + \beta(rq_x - qr_x) + \tfrac{1}{2}\beta(8\beta - 3)q^2r^2$$
$$v_{12} = 2q\lambda^3 + [-q_x + (2\beta - 1)q^2r]\lambda \qquad v_{21} = 2r\lambda^3 + [r_x + (2\beta - 1)qr^2]\lambda.$$

Example 2. The second system of the hierarchy (19) ($n = 2$) is

$$\begin{aligned} q_t - \tfrac{1}{4}[&2q_{xxx} - 6(2\beta - 1)rq_x^2 - 6(4\beta - 1)qq_xr_x - 6(2\beta - 1)qrq_{xx} \\ &+6(2\beta - 1)(4\beta - 1)q^3rr_x + 3(8\beta^2 - 12\beta + 3)q^2r^2q_x \\ &+4\beta(2\beta - 1)(4\beta - 1)q^4r^3] = 0 \\ r_t - \tfrac{1}{4}[&2r_{xxx} + 6(2\beta - 1)qr_x^2 - 6(4\beta - 1)rq_xr_x + 6(2\beta - 1)qrr_{xx} \\ &+6(2\beta - 1)(4\beta - 1)qr^3q_x + 3(8\beta^2 - 12\beta + 3)q^2r^2r_x \\ &-4\beta(2\beta - 1)(4\beta - 1)q^3r^4] = 0 \end{aligned} \tag{24}$$

which reduces to the MKdV equation for $r = 1$, $\beta = 1/2$

$$q_t - \tfrac{1}{4}(2q_{xxx} - 3q^2q_x) = 0$$

and the STO equation [19–21] for $r = 1$, $\beta = 1/4$

$$q_t - \tfrac{1}{8}(4q_{xxx} + 6q_x^2 + 6qq_{xx} + 3q^2q_x) = 0.$$

The system (24) is Liouville integrable and possesses the tri-Hamiltonian structure

$$u_t = J\frac{\delta H_2}{\delta u} = JL\frac{\delta H_1}{\delta u} = JL^2\frac{\delta H_0}{\delta u}$$

where H_0 and H_1 are defined by (22), and

$$H_2 = \tfrac{1}{8}[qr_{xx} + rq_{xx} + (8\beta - 3)(q^2rr_x - r^2qq_x) + 2(2\beta - 1)(4\beta - 1)q^3r^3].$$

The Lax pairs corresponding to the system (24) are given by spectral problem (1) and the auxiliary problem

$$y_t = V^{(2)}y = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & -v_{11} \end{pmatrix} y$$

with

$$v_{11} = -2\lambda^6 + qr\lambda^4 + \tfrac{1}{4}[2(qr_x - rq_x) + (8\beta - 3)q^2r^2]\lambda^2 + 2\beta a_6$$
$$v_{12} = 2q\lambda^5 + [-q_x + (2\beta - 1)q^2r]\lambda^3 + b_5\lambda$$
$$v_{21} = 2r\lambda^5 + [r_x + (2\beta - 1)qr^2]\lambda^3 + c_5\lambda$$

where a_6, b_5 and c_5 are given by (7)–(9).

In summary, starting from spectral problem (1), we derive a hiearchy of DNLS type equations (19), which is actually a generalization of KN hierarchy ($\beta = 0$) [1, 7, 8]. More importantly, the hierarchy (19) contains five kinds of well known equation (KN, CLL, GI, MKdV and STO equations). Among them only the KN equation belongs to the KN hierarchy. It is also interesting that the KN, CLL and GI equations can be expressed by a generalized DNLS equation involving a parameter. Their Hamiltonian structure and Lax pairs are also given by unified and explicit formulae.

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